JOURNAL OF APPROXIMATION THEORY 3, 410–429 (1970)

Saturation of Positive Convolution Operators

RONALD A. DEVORE

Department of Mathematics, Oakland University, Rochester, Michigan 48036

Received December 4, 1969

INTRODUCTION

$C^*[-\pi, \pi]$ denotes the space of 2π-periodic continuous functions with the usual supremum norm $\|\cdot\|$. Let $(L_n)_{n=1}^{\infty}$ be a sequence of linear operators on $C^*[-\pi, \pi]$, given by the convolution formula

$$L_n(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t)\, d\mu_n(t) = f * d\mu_n\,, \tag{1.1}$$

where $d\mu_n$ is an even Borel measure on $[-\pi, \pi]$, with

$$\frac{1}{\pi} \int_{-\pi}^{\pi} d\mu_n(t) = 1, \qquad n = 1, 2,...$$

The sequence (L_n) is said to be saturated if there is a sequence of positive numbers, $(\phi(n))$, converging to 0 such that

$$\|f - L_n(f)\| = o(\phi(n)) \text{ if and only} \tag{1.2}$$

if f is a constant, and
there is a non-constant function f_0 such that

$$\|f_0 - L_n(f_0)\| = O(\phi(n)). \tag{1.3}$$

The sequence $(\phi(n))$ is called the saturation order of (L_n). Define the saturation class $S(L_n)$ to be the set of all functions f in $C^*[-\pi, \pi]$ such that

$$\|f - L_n(f)\| = O(\phi(n)). \tag{1.4}$$

An excellent source for references on saturation theorems is the paper of Butzer and Görlich [10] or the book [11]. We quote here the following well-known theorem of Sunouchi-Watari [1].

For $f \in C^*[-\pi, \pi]$ we let

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \bar{e}^{ikt} f(t)\, dt \qquad k = 0, \pm 1, \pm 2, \ldots$$

and

$$\rho_k^{(n)} = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-ikt}\, d\mu_n(t)\, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos kt\, d\mu_n(t) \qquad k = 0, \pm 1, \pm 2, \ldots .$$

The Sunouchi-Watari Theorem can then be stated as

THEOREM A. *Suppose* $0 < \phi(n) \to 0$ *and*

$$\psi_k = \lim_{n \to \infty} \frac{1 - \rho_k^{(n)}}{\phi(n)} \tag{1.5}$$

exists and is different from $0, \infty, -\infty$, *for* $k = \pm 1, \pm 2, \ldots$. *Then* (L_n) *is saturated with order* $(\phi(n))$ *and if* f *is in* $S(L_n)$ *then*

$$\sum_{-\infty}^{\infty} \psi_k \hat{f}(k)\, e^{ikx} \qquad \text{is in } L_\infty . \tag{1.6}$$

Also, if, for $n = 1, 2, \ldots$, $(1 - \rho_k^{(n)}) / \psi_k \phi(n))_{k=-\infty}^{\infty}$ *are multipliers from* L_∞ *to* L_∞, *uniform in n, then* (1.6) *is sufficient for* f *to be in* $S(L_n)$.

In this paper we shall impose the additional hypothesis that the measures $d\mu_n$ be positive. This hypothesis is indeed a strong one, in light of the many striking results for positive operators due to Korovkin [6] and others. However, it is also a very natural and important condition since many of the classical linear methods of approximation are of this type.

For sequences of positive convolution operators, it is not difficult to characterize precisely when they are saturated. This is done in Section 3. The characterization of the class $S(L_n)$ is much more difficult. A result in this direction is the following theorem of Tureckii [2] which we shall discuss further later.

THEOREM B. *Suppose* (L_n) *is a sequence of positive convolution operators and*

$$\lim_{n \to \infty} \frac{1 - \rho_k^{(n)}}{1 - \rho_1^{(n)}} = k^2, \qquad k = \pm 1, \pm 2, \ldots .$$

Then (L_n) *is saturated with order* $(1 - \rho^{1(n)})$ *and* $S(L_n) = \{f : f' \in \text{Lip } 1\}$.

In Section 4 we develop some new techniques which are useful in characterizing the saturation class $S(L_n)$ for certain sequences of positive convolution operators. Our main application of these techniques is to optimal and quasi-optimal sequences of trigonometric operators to be defined below.

The sequence $(L_n)_{n=1}^{\infty}$ is said to be a sequence of trigonometric operators if for each $f \in C^*[-\pi, \pi]$, $L_n(f)$ is a trigonometric polynomial of degree $\leqslant n$, $n = 1, 2, \ldots$. In case of convolution operators, this is equivalent to having $d\mu_n(t) = T_n(t)\,dt$, where T_n is a non-negative even trigonometric polynomial of degree $\leqslant n$. The Fejér [3], Jackson [4] and Korovkin [5] operators are the best known operators of this type.

It was first noted by Korovkin [6] that positive trigonometric operators are limited in their efficiency of approximation. Korovkin showed that if (L_n) is a sequence of positive trigonometric operators, then for g one of the three functions 1, cos x, sin x,

$$\| g - L_n(g) \| \neq o\left(\frac{1}{n^2}\right). \tag{1.7}$$

In the case of positive trigonometric convolution operators, this is equivalent to

$$\int_{-\pi}^{\pi} \sin^2 \frac{t}{2}\, T_n(t)\, dt \neq o\left(\frac{1}{n^2}\right). \tag{1.8}$$

We shall say that (L_n) is an optimal sequence if

$$\int_{-\pi}^{\pi} \sin^2 \frac{t}{2}\, T_n(t)\, dt = O\left(\frac{1}{n^2}\right). \tag{1.9}$$

If we ask which sequences of positive trigonometric convolution operators (L_n) give the Jackson estimate

$$\| f - L_n(f) \| \leqslant \frac{C}{n}\, \omega\left(f', \frac{1}{n}\right) \tag{1.10}$$

for all functions f with $f' \in C^*[-\pi, \pi]$, the answer is: exactly the optimal sequences (see Theorem 2.1). Thus, for example, the Jackson and Korovkin operators are optimal whereas the Fejér operators are not. Curtis [7] has shown that if (L_n) is a sequence of positive trigonometric convolution operators then

$$\| f - L_n(f) \| = o\left(\frac{1}{n^2}\right) \tag{1.11}$$

if and only if f is a constant. This can be considered as a strengthening of

Korovkin's Theorem (1.7). In Section 2, it is shown that if (L_n) is an optimal sequence then for each f such that $f' \in \mathrm{Lip}\ 1$,

$$\|f - L_n(f)\| \leqslant \frac{C}{n^2}, \qquad n = 1, 2, \ldots .$$

Thus, there are always nonconstant functions f which are approximated by the sequence $(L_n(f))$ with order $O(n^{-2})$. This, together with (1.11) shows that optimal sequences are always saturated with order n^{-2}.

At the recent Symposium on the Constructive Theory of Functions held in Budapest, the author has conjectured that the saturation class $S(L_n)$ of every optimal sequence is $\{f : f' \in \mathrm{Lip}\ 1\}$. This is already known for the Jackson and Korovkin operators by virtue of Theorem B. Our main result is a proof of this conjecture (Theorem 5.1).

At the same conference, Görlich [9] suggested considering sequences L_n for which there exists a nonconstant function f_0 such that

$$\|f_0 - L_n(f_0)\| = O\left(\frac{1}{n^2}\right).$$

We shall call such sequences quasi-optimal. That quasi-optimal sequences are also saturated with order n^{-2} follows from (1.11). Görlich has asked whether quasi-optimal sequences also have as a saturation class $\{f : f' \in \mathrm{Lip}\ 1\}$. It is easy to construct an example of a sequence (L_n) which is quasi-optimal but not optimal. For example, if $K_n(t)$ is the Jackson kernel (see Section 3) then the trigonometric polynomials

$$T_n(t) = \frac{n}{n+2}\left(K_n(t) + \frac{1}{n}\left[K_n(t+\pi) + K_n(t-\pi)\right]\right)$$

generate by convolution a sequence $(L_n{}^*)$ of operators which is quasi-optimal but not optimal. It is not optimal since there is a constant $C > 0$ such that

$$\int_{-\pi}^{\pi} \sin^2 \frac{t}{2}\, T_n(t)\, dt \geqslant \frac{C}{n}, \qquad n = 1, 2, \ldots . \tag{1.12}$$

That it is quasi-optimal follows from Corollary 2.1 and the fact that there exists a constant $C' > 0$ such that

$$\int_{-\pi}^{\pi} \sin^2 t T_n(t)\, dt \leqslant \frac{C'}{n^2}, \qquad n = 1, 2, \ldots . \tag{1.13}$$

The inequalities (1.12) and (1.13) can be used to show that if

$$\|f - L_n{}^*(f)\| = O(n^{-2}), \quad \text{then } f \text{ has period } \pi.$$

The saturation class of $(L_n{}^*)$ is $\{f : f' \in \text{Lip } 1 \text{ and } f \text{ has period } \pi\}$

This is a special case of the general result we obtain for quasi-optimal sequences. In Section 5 we show that if (L_n) is quasi-optimal, then there is an integer m such that

$$\int_{-\pi}^{\pi} \sin^2 \frac{mt}{2} T_n(t)\, dt = O\left(\frac{1}{n^2}\right). \tag{1.13}$$

When $m = 1$, we have an optimal sequence. If m_0 is the smallest positive integer such that (1.13) holds, then, as we show, the saturation class is $\{f : f' \in \text{Lip } 1, f \text{ has period } (2\pi/m_0)\}$.

2. A Preliminary Inequality

There are inequalities which estimate the rate of convergence of a sequence $(L_n(f))$ to f, for continuous functions f, in terms of the rate of convergence of $(L_n(g))$ to g, for the three functions g : 1, $\cos x$ and $\sin x$. The most general of these estimates is the inequality of Shisha and Mond [8]. We shall need an extension of this inequality which estimates $\|f - L_n(f)\|$ for functions having period $2\pi/m$, in terms of the behavior of $\|g - L_n(g)\|$ for the functions g : 1, $\cos mx$, $\sin mx$. We shall use this extension only for convolution operators; However, we shall prove it for the general case of arbitrary linear positive operators.

THEOREM 2.1. *Let m be a positive integer and (L_n) a sequence of linear positive operators. If $0 \leqslant \lambda_n \to 0$ satisfies*

$$\|g - L_n(g)\| = O(\lambda_n{}^2) \tag{2.1}$$

for the three functions g : 1, $\cos mx$ *and* $\sin mx$*, then for each continuous f with period $2\pi/m$ we have*

$$\|f - L_n(f)\| \leqslant C(1 + \|f\|)(\lambda_n + \omega(f, \lambda_n)). \tag{2.2}$$

If, in addition, f is everywhere continuously differtiable, then

$$\|f - L_n(f)\| \leqslant C(1 + \|f\| + \|f'\|)(\lambda_n{}^2 + \lambda_n \omega(f', \lambda_n)) \tag{2.3}$$

where C is a constant independent of f and n, and ω denotes the modulus of continuity.

Proof. The proofs of (2.2) and (2.3) are similar and we shall only prove (2.3).

Writing $\sin m(x - x_0)$ and $\sin^2 (m(x - x_0)/2)$ in terms of $\sin mx$ and $\cos mx$, one can easily deduce from (2.1) that

$$\| L_n(\sin m(t - x), x)\| = O(\lambda_n^2), \tag{2.4}$$

$$\left\| L_n \left(\sin^2 m \left(\frac{t-x}{2}\right), x\right)\right\| = O(\lambda_n^2). \tag{2.5}$$

Note. Equations (2.4) means that L_n is applied to $\sin m(t - x)$ as a function of t (with x fixed). The result is evaluated at x, and finally, we take the norm of the resulting function of x. Such a convention is used throughout.

Let f be continuously differentiable and of period $2\pi/m$. In what follows, C will always denote some constant independent of f and n. We shall first show that

$$\begin{aligned} &\left\| L_n \left(\left| f(t) - f(x) - \frac{f'(x)}{m} \sin m(t-x)\right|, x\right)\right\| \\ &\qquad \leqslant C(1 + \|f'\|)(\lambda_n^2 + \lambda_n \omega(f', \lambda_n)). \end{aligned} \tag{2.6}$$

Indeed, if $x \in [-\pi, \pi]$ then

$$\begin{aligned} &\left| f(t) - f(x) - \frac{f'(x)}{m} \sin m(t-x)\right| \\ &\quad = \left| f(t) - f(x) - f'(x)(t-x) + f'(x)\left[(t-x) - \frac{\sin m(t-x)}{m}\right]\right| \\ &\quad \leqslant | t - x | \, \omega(f', | t - x |) + C \|f'\| \cdot | t - x |^3. \end{aligned}$$

If $| t - x | \leqslant \pi/m$ then $| t - x | \leqslant \pi/m \sin m(| t - x |/2)$. Using elementary properties of ω, we have

$$\begin{aligned} &\left| f(t) - f(x) - \frac{f'(x)}{x} \sin m(t-x)\right| \\ &\quad \leqslant C\left[\sin m \frac{| t - x |}{2} \, \omega\left(f', \sin m \frac{| t - x |}{2}\right) + \|f'\| \sin^3 \frac{| t - x |}{2}\right]. \end{aligned} \tag{2.7}$$

Now,

$$\begin{aligned} \omega\left(f', \sin m \frac{| t - x |}{2}\right) &= \omega\left(f', \frac{\lambda_n}{\lambda_n} \sin m \frac{| t - x |}{2}\right) \\ &\leqslant \left(1 + \lambda_n^{-1} \sin m \frac{| t - x |}{2}\right) \omega(f', \lambda_n). \end{aligned}$$

thus, replacing $\sin^3 m(|t-x|/2)$ by the larger number $\sin^2 m(|t-x|/2)$ in (2.7), we have

$$\left|f(t)-f(x)-\frac{f'(x)}{m}\sin m(t-x)\right|$$
$$\leqslant C(1+\|f'\|)\Big[\sin^2 m\frac{(t-x)}{2}+\sin^2 m\frac{|t-x|}{2}\frac{\omega(f',\lambda_n)}{\lambda_n}$$
$$+\sin m\frac{|t-x|}{2}\omega(f',\lambda_n)\Big]. \tag{2.8}$$

Since the functions in (2.8) have period $2\pi/m$, this estimate holds for all t and x. Applying L_n to (2.8), we have

$$L_n\left(|f(t)-f(x)|-\frac{f'(x)}{m}\sin m|t-x|\right)$$
$$\leqslant C(1+\|f'\|)(\lambda_n^2+\omega(f',\lambda_n)\left(\lambda_n+\left\|L_n\left(\sin m\frac{|t-x|}{2},x\right)\right\|\right). \tag{2.9}$$

The Cauchy-Schwarz inequality for L_n gives

$$\left\|L_n\left(\sin m\frac{|t-x|}{2},x\right)\right\|\leqslant\|L_n(1)\|^{1/2}\left\|L_n\left(\sin^2 m\frac{(t-x)}{2},x\right)\right\|^{1/2}\leqslant C\lambda_n,$$

which, when substituted into (2.9), gives (2.6).
Finally, to obtain (2.3), we write

$$f(x)-L_n(f(t),x)$$
$$=f(x)-L_n(f(x),x)+L_n\left(f(t)-f(x)-\frac{f'(x)}{m}\sin m(t-x),x\right)$$
$$+\frac{f'(x)}{m}L_n(\sin m(t-x),x).$$

Thus, from (2.6),

$$\|f-L_n(f)\|\leqslant\|f(x)-L_n(f(x),x)\|+C(1+\|f'\|)(\lambda_n^2+\omega(f',\lambda_n))$$
$$+\frac{\|f'\|}{m}\|L_n(\sin m(t-x),x)\|.$$

Estimating the first and last terms of the right hand side by (2.1) and (2.4), gives the desired result (2.3) and the theorem is proved.

Theorem 2.1 can be stated in the following form which is more convenient for later applications.

COROLARRY 2.1. *If $L_n(f) = f * d\mu_n$, where $d\mu_n$ is a non-negative even Borel measure on $[-\pi, \pi]$ such that*

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\mu_n(t) = 1 \tag{2.10}$$

and

$$\int_{-\pi}^{\pi} \sin^2\frac{mt}{2}\, d\mu_n(t) = O(\lambda_n^2) \qquad (n \to \infty), \tag{2.11}$$

then for each f of period $2\pi/m$ we have

$$\|f - L_n(f)\| \leqslant C(1 + \|f\|)(\lambda_n + \omega(f, \lambda_n)), \quad n = 1, 2,... \tag{2.12}$$

when f is continuous, and

$$f - L_n(f)\| \leqslant C(1 + \|f\| + \| f' \|)(\lambda_n^2 + \lambda_n\omega(f', \lambda_n)), \quad n = 1, 2,..., \tag{2.13}$$

when f is continuously differentiable.

Proof. Since $d\mu_n$ is even, we have

$$\int_{-\pi}^{\pi} \sin mt\, d\mu_n(t) = 0, \qquad n = 1, 2,... . \tag{2.15}$$

Integrating the identity

$$\sin mt - \sin mx = \sin m(t - x) \cos mx - (1 - \cos m(t - x)) \sin mx,$$

we obtain

$$\left|\frac{1}{\pi}\int_{-\pi}^{\pi} (\sin mt - \sin mx)\, d\mu_n(t - x)\right| \leqslant \frac{2}{\pi}\int_{-\pi}^{\pi} \sin^2\frac{mt}{2}\, d\mu_n(t) \leqslant C\lambda_n^2,$$
$$n = 1, 2,...,$$

i.e.,

$$\| \sin mx - L_n(\sin mt, x)\| \leqslant C\lambda_n^2, \qquad n = 1, 2,....$$

In a similar manner, we obtain

$$\| \cos mx - L_n(\cos mt, x)\| \leqslant C\lambda_n^2, \qquad n = 1, 2,...,$$

and, thus the hypotheses of Theorem 2.1 are verified.

3. Saturation of Positive Convolution Operators

In the remainder of the paper, we shall assume L_n has the form

$$L_n(f) = f * d\mu_n ,$$

where $d\mu_n$ is an even non-negative Borel measure on $[-\pi, \pi]$ with $1/\pi \int_{-\pi}^{\pi} d\mu_n = 1$.

The following theorem determines when (L_n) is saturated and, when it is, its saturation order.

THEOREM 3.1. *The sequence (L_n) is saturated if and only if there is a positive integer m such that*

$$\varlimsup_{n\to\infty} \frac{1-\rho_k^{(n)}}{1-\rho_m^{(n)}} \geqslant C_k > 0, \qquad k = 1, 2,..., \tag{3.1}$$

where C_k may be infinity. When (L_n) is saturated, the saturation is $(1 - \rho_m^{(n)})$ *where m is any integer satisfying* (3.1)

Proof. We first suppose that (3.1) holds. If $f \in C^*[-\pi, \pi]$ and

$$\|f - L_n(f)\| = o(1 - \rho_m^{(n)}) \qquad (n \to \infty)$$

then

$$\int_{-\pi}^{\pi} (f[x] - (f * d\mu_n)[x])\, e^{ikx}\, dx = o(1 - \rho_m^{(n)}) \qquad k = \pm 1, \pm 2,...,$$

i.e.,

$$\hat{f}(k) \frac{(1 - \rho_k^{(n)})}{(1 - \rho_m^{(n)})} = o(1), \qquad k = \pm 1, \pm 2,... .$$

Therefore

$$C_k \hat{f}(k) = 0, \qquad k = \pm 1, \pm 2,...,$$

and, thus, $\hat{f}(k) = 0$, $k = \pm 1, \pm 2,...$ so that f is a constant. Also, for each f of period $2\pi/m$ such that $f' \in \operatorname{Lip} 1$, Corollary 2.1 gives

$$\|f - L_n(f)\| = O(1 - \rho_m^{(n)}).$$

To prove the converse, suppose that (L_n) is saturated with order (λ_n). Assume first, that for each $k = \pm 1, \pm 2, \ldots$,

$$\overline{\lim_{n\to\infty}} \frac{1 - \rho_k^{(n)}}{\lambda_n} = \infty. \tag{3.2}$$

Then, if $f \in C^*[-\pi, \pi]$ and

$$\|f - L_n(f)\| = O(\lambda_n), \tag{3.3}$$

we have

$$\hat{f}(k) \frac{(1 - \rho_k^{(n)})}{\lambda_n} = O(1), \qquad k = \pm 1, \pm 2, \ldots,$$

which, in view of (3.2), implies $\hat{f}(k) = 0$, $k = \pm 1, \pm 2, \ldots$, i.e., f is a constant. Thus, there are no non-constant functions satisfying (3.3), contradicting our hypothesis that (L_n) is saturated with order (λ_n). Thus, there must exist an integer m such that

$$\overline{\lim} \frac{1 - \rho_m^{(n)}}{\lambda_n} < +\infty.$$

Suppose there is an integer k_0 such that

$$\underline{\lim_{n\to\infty}} \frac{1 - \rho_{k_0}^{(n)}}{1 - \rho_m^{(n)}} = 0.$$

Then from inequality (2.3) of Theorem 2.1 we have that for each function f of period $2\pi/k_0$, with $f' \in \operatorname{lip} 1$,

$$\|f - L_n(f)\| = O(1 - \rho_{k_0}^{(n)}) = o(1 - \rho_m^{(n)}) \qquad (n \to \infty),$$

a contradiction; the theorem is proved.

It is easy to give examples of operators L_n for which (L_n) is not saturated. We give now such an example where L_n is of the particularly simple form $L_n(f) = f * T_n$, with T_n a non-negative even trigonometric polynomial of degree $2n - 2$. This example is particularly interesting in view of a theorem of Curtis [7] stating that for any sequence of trigonometric convolution operators (L_n), $\|f - L_n(f)\| = o(n^{-2})$ if and only if f is constant. This theorem suggests that perhaps all such sequences are saturated.

Let K_n denote the Jackson kernel

$$K_n(t) = C_n \left(\frac{\sin nt/2}{\sin t/2}\right)^4,$$

with C_n chosen so that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t)\,dt = 1.$$

We shall need the following well-known properties of $K_n(t)$ (see [3]):

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2\frac{mt}{2} K_n(t)\,dt = \frac{3}{2}\frac{m^2}{n^2} + o(1/n^2) \qquad (m = 1, 2,\ldots). \tag{3.4}$$

$$\text{If} \quad f \in C^*[-\pi, \pi] \qquad \text{then} \quad \|f * K_n - f\| \to o. \tag{3.5}$$

THEOREM 3.2. *For $n = 1, 2,\ldots,$ let*

$$\Lambda_n(t) = a_n \left(K_n(t) + \sum_{k=1}^{\infty} \frac{1}{n^{2-1/k}} \cdot \frac{1}{k^2} [K_n(t - x_k) + K(t + x_k)]\right),$$

where

$$a_n^{-1} = 1 + 2\sum_{k=1}^{\infty} \frac{1}{k^2 n^{1-1/k}} \qquad \textit{and} \qquad x_k = \frac{\pi}{2^k}, \quad k = 1, 2,\ldots .$$

Then the sequence (L_n), given by

$$L_n(f) = f * \Lambda_n,$$

is not saturated.

Proof. Let k be any integer, $k = 2^m j$ with j not divisible by 2, then

$$1 - 2\hat{\Lambda}_n(2k) = 1 - 2\hat{\Lambda}_n(2^{m+1}j) = 2\int_{-\pi}^{\pi} \sin(2^m jt)\,\Lambda_n(t)\,dt.$$

But for any m and j,

$$\int_{-\pi}^{\pi} \sin^2(2^m jt)\,\Lambda_n(t)\,dt$$

$$= \int_{-\pi}^{\pi} \sin^2(2^m jt)\,K_n(t)\,dt + 2\sum_{k=1}^{\infty}\frac{1}{k^2 n^{2-1/k}} \int_{-\pi}^{\pi} \sin^2(2^m jt)\,K_n(t - x_k)\,dt$$

since, for each n, the series converges uniformly in t.

By virtue of (3.4) and the fact that $\sin^2(2^m[t - x_k]) = \sin^2(2^m t)$ for $k \leqslant m$,

$$\left| \int_{-\pi}^{\pi} \sin^2(2^m jt)\, dt + 2 \sum_{k=1}^{m} \frac{1}{k^2 n^{2-1/k}} \int_{-\pi}^{\pi} \sin^2(2^m jt)\, K_n(t - x_k)\, dt \right|$$

$$\leqslant \frac{3}{2}\left(\frac{4^m}{n^2} + 2 \sum_{k=1}^{m} \frac{4^m}{n^2 k^2 n^{2-1/k}}\right) + o\left(\frac{1}{n^3}\right) = o\left(\frac{1}{n^{2-1/m+1}}\right).$$

Also,

$$\left| \sum_{k=m+2}^{\infty} \frac{1}{k^2 n^{2-1/k}} \int_{-\pi}^{\pi} \sin^2(2^m jt)\, K_n(t - x_k)\, dt \right|$$

$$\leqslant \sum_{k=m+2}^{\infty} k^{-2} n^{(1/k)-2} = o(n^{(m+1)^{-1}-2}).$$

However, by virtue of (3.5),

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \sin^2(2^m jt)\, K_n(t - x_{m+1})\, dt = \sin^2 \frac{\pi}{2} = 1.$$

Thus,

$$n^{2-1/m+1}(1 - 2\hat{\Lambda}_n(2^{m+1}j)) = 2 + o(1), \qquad m = 1, 2, \ldots,$$

therefore

$$1 - 2\hat{\Lambda}_n(2k) = 0(1 - 2\hat{\Lambda}_n(k))$$

and, therefore, by Theorem 3.1, (L_n) is not saturated.

4. Saturation classes of sequences of positive convolution operators

In this section, we wish to examine the saturation class $S(L_n)$ of the sequence (L_n). The theorem of Sunouchi-Watari (Theorem A) applies to general convolution operators and hence, to (L_n). Our first objective is to extend this theorem, for positive operators, to the case where the m-th Fourier coefficients determine the saturation order. We first wish to show that when the m-th coefficients determine the saturation order, every function in $S(L_n)$ has period $2\pi/m$.

Lemma 4.1. *Let (L_n) be saturated and let m denote the smallest positive integer such that* (3.1) *is satisfied. If f is in $S(L_n)$ then it has period $2\pi/m$.*

Proof. For each $0 < k < m$, $\overline{\lim}_{n\to\infty} (1 - \rho_k^{(n)}/1 - \rho_m^{(n)}) = \infty$. Similarly, suppose k is an integer $> m$ but not divisible by m, say, $k = lm + j$, $o < j < m$, $1 \geqslant 1$. Then for some positive constant C,

$$C\left(\sin^2 \frac{mx}{2} + \sin^2 \frac{(lm+j)x}{2}\right) \geqslant \sin^2 \frac{jx}{2}$$

since every zero of the left hand side is a zero of the right hand side with the same multiplicity. Thus,

$$C\int_{-\pi}^{\pi} \sin^2 \frac{mx}{2}\, d\mu_n(x) + C\int_{-\pi}^{\pi} \sin^2 \frac{(lm+j)x}{2}\, d\mu_n(x) \geqslant \int_{-\pi}^{\pi} \sin^2 \frac{jx}{2}\, d\mu_n(x),$$

i.e.,

$$C + \frac{C(1-\rho_k^{(n)})}{1-\rho_m^{(n)}} \geqslant \frac{1-\rho_j^{(n)}}{1-\rho_m^{(n)}}$$

and, therefore,

$$\overline{\lim}\, \frac{1-\rho_k^{(n)}}{1-\rho_m^{(n)}} = +\infty.$$

Now, if $f \in S(L_n)$ then

$$f - f * d\mu_n = O(1-\rho_m^{(n)})$$

so that

$$\hat{f}(k)(1-\rho_k^{(n)}) = O(1-\rho_m^{(n)}), \qquad k = \pm 1, \pm 2, \ldots.$$

If $k \not\equiv O \pmod m$ we must have $\hat{f}(k) = 0$ which establishes the Lemma.

We state now our generalization of the Sunouchi-Watari Theorem.

THEOREM 4.1. *If (L_n) is saturated with order $(1-\rho_m^{(n)})$, where m is the least positive integer satisfying* (3.1) *and if*

$$\psi_j = \lim_{n\to\infty} \frac{1-\rho_{jm}^{(n)}}{1-\rho_m^{(n)}} \qquad \textit{exists and is finite for} \quad j = \pm 1, \pm 2, \ldots, \tag{4.1}$$

then if f is in $S(L_n)$, only if it has period $2\pi/m$ and
* $\sum_{-\infty}^{\infty} \psi_j \hat{f}(jm)\, e^{ijmx}$ *is the Fourier series of a function in L_∞. If, in addition to* (4.1), *for $n = 1, 2, \ldots$,*

$$\left(\frac{1-\rho_{jm}^{(n)}}{1-\rho_m^{(n)}}\right)_{j=-\infty}^{\infty}$$

are multipliers from L_∞ to L_∞, uniform in n, then (*) *is also sufficient for f to be in $S(L_n)$.*

The proof of this theorem is the same as that of Theorem A and will be ommitted.

The theorem analogous to Theorem B is the following

THEOREM 4.2. *If* (L_n) *is saturated with order* $(1 - \rho_m^{(n)})$, *where m is the least positive integer satisfying* (3.1) *and if*

$$\lim_{n\to\infty} \frac{1 - \rho_{jm}^{(n)}}{1 - \rho_m^{(n)}} = j^2, \qquad j = \pm 1, \pm 2, \ldots, \tag{4.2}$$

then $S(L_n) = \{f : f' \in \mathrm{Lip}\ 1, f \text{ has period } 2\pi/m\}$.

Again, the proof is the same as that of Theorem B and will be ommitted.

Although Theorems A and B have wide application to classical methods of approximation and, in particular, are applicable to the operators of Fejér, Jackson and Korovkin, they cannot be applied to the general case of optimal or quasi-optimal sequences since condition (4.1) need not be satisfied. Therefore, we wish now to take a different approach to the problem of determining $S(L_n)$ which will handle the cases of optimal and "quasi-optimal timal sequences."

The class $S(L_n)$ is precisely the class of $2\pi/m$-periodic functions for which

$$\left\| \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\frac{f(x+t) + f(x-t) - 2f(x)}{\sin^2 mt/2} \right] (1 - \rho_m^{(n)})^{-1} \sin^2 \frac{mt}{2}\, d\mu_n(t) \right\| = O(1).$$

Let $d\lambda_n(t) = (1 - \rho_m^{(n)})^{-1} (\sin^2 mt/2)\, d\mu_n(t)$. Since

$$\frac{1}{\pi} \int_{-\pi}^{\pi} | d\lambda_n | = \frac{1}{2}, \qquad n = 1, 2, \ldots,$$

$\{d\lambda_n\}_{n=1}^{\infty}$ is contained in a compact subset of the dual of $C^*[-\pi, \pi]$, in the weak* topology. Therefore, a subsequence $(d\lambda_{n_k})$ converges weak * to some measure $d\lambda$. If such a dλ is $\alpha d\rho_0$, where $d\rho_0$ is the Dirac measure at zero, then it is easy to see that $S(L_n) = \{f : f' \in \mathrm{Lip}\ 1, f \text{ has period } 2\pi/m\}$. The following theorem strengthens this result by requiring only that $d\lambda$ has an atom at one of the zeros of $\sin^2 mt/2$.

THEOREM 4.3. *If* $d\lambda$ *is a weak * limit of some subsequence* $(d\lambda_{n_k})$ *and* $d\lambda$ *has an atom at one of the zeros of* $\sin^2 mt/2$ *then* $S(L_n) = \{f : f' \in \mathrm{Lip}\ 1, f \text{ has period } 2\pi/m\}$.

Proof. Let x_k $(k = 1, 2, \ldots)$ be the distinct zeros of $\sin^2 mt/2$ in $[-\pi, \pi]$

and, for every $\epsilon > 0$, let $S_\epsilon = \bigcup_k (x_k - \epsilon, x_k + \epsilon)$, $T_\epsilon = S_\epsilon \backslash \{x_1, x_2, ...\}$. Let $\alpha = \lim_{\epsilon \to 0} \int_{S_\epsilon} d\lambda(t) > 0$ and choose $0 < \delta < \pi/m$ such that

$$\frac{\pi^2}{2} \int_{T_\delta} d\lambda(t) < \alpha. \tag{4.3}$$

If h is a twice continuously differentiable function of period $2\pi/m$, then for $x_0 \epsilon [-\pi, \pi]$, the function

$$g(t) = \frac{h(x_0 + t) + h(x_0 - t) - 2h(x_0)}{\sin^2 mt/2}$$

is continuous on $[-\pi, \pi]$ and its value at $x_k (k = 1, 2, ...)$ is $4m^{-2} h''(x_0)$. Suppose $\| h - L_n(h) \| \leqslant M(1 - \rho_m^{(n)})$, $n = 1, 2, ...$ Then

$$\frac{1}{\pi} \left| \int_{-\pi}^{\pi} g(t)\, d\lambda(t) \right| = \lim_{k \to \infty} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} g(t)\, d\lambda_{n_k}(t) \right| \leqslant M,$$

M being a constant. Therefore,

$$\frac{4\alpha}{\pi m^2} \mid h''(x_0) \mid \leqslant \left| \frac{1}{\pi} \int_{T_\delta} g(t)\, d\lambda(t) \right| + \left| \frac{1}{\pi} \int_{[-\pi, \pi] - S_\delta} g(t)\, d\lambda(t) \right| + M. \tag{4.4}$$

We estimate the first summand on the right hand side of (4.4) as follows. For $| t | \leqslant \delta < \pi/m$, $\sin^2 mt/2 \geqslant m^2 t^2/\pi^2$ and, thus,

$$\left| \frac{h(x_0 + t) + h(x_0 - t) - 2h(x_0)}{\sin^2 mt/2} \right| \leqslant \frac{\pi^2}{m^2} \left| \frac{h(x_0 + t) + h(x_0 - t) - 2h(x_0)}{t^2} \right|.$$

By the periodicity of $g(t)$ we have

$$| g(t) | \leqslant \frac{\pi^2}{m^2} \| h'' \|, \qquad \text{for} \quad t \in T_\delta. \tag{4.5}$$

For the second summand on the right-hand side of (4.4) we have, for $\delta \leqslant | t | \leqslant 2\pi/m - \delta$,

$$\left| \frac{h(x_0 + t) + h(x_0 - t) - 2h(x_0)}{\sin^2 mt/2} \right| \leqslant \frac{4 \| h \|}{\sin^2 m\delta/2},$$

so that, using again the periodicity of h,

$$\left| \frac{1}{\pi} \int_{[-\pi, \pi] - S_\delta} g(t)\, d\lambda(t) \right| \leqslant \frac{4 \| h \|}{\pi \sin^2 m\delta/2} \int_{-\pi}^{\pi} | d\lambda | = C_\delta \| h \|. \quad -(4.6)$$

Using (4.5) and (4.6) in (4.4), we have

$$\frac{4\alpha}{\pi m^2} \mid h''(x_0) \mid \leqslant \frac{\pi^2}{m^2} \| h'' \| \left(\frac{1}{\pi} \int_{T_\delta} d\lambda(t) \right) + C_\delta \| h \| + M.$$

Taking into account (4.3), we obtain

$$\| h'' \| \leqslant \frac{m^2 \pi}{2\alpha} (C_\delta \| h \| + M). \tag{4.7}$$

Now, let f be in $S(L_n)$ with $\|f - L_n(f)\| \leqslant M(1 - \rho_m^{(n)})$ $n = 1, 2, \ldots$. If p is a positive integer and $K_p(t)$ is the corresponding Jackson kernel the function $f * K_p$ is a trigonometric polynomial of period $2\pi/m$. Also

$$\begin{aligned} \|f * K_p - L_n(f * K_p)\| &= \|(f - f * d\mu_n) * K_p \| \leqslant \|f - f * d\mu_n \| \int_{-\pi}^{\pi} | K_p | \\ &\leqslant M(1 - \rho_m^{(n)}), \qquad n, p = 1, 2, \ldots . \end{aligned}$$

Therefore, from (4.7) we obtain

$$\|(f * K_p)''\| \leqslant \frac{m^2 \pi}{2\alpha} (C_\delta \|f * K_p \| + M) \leqslant C,$$

where C is a constant independent of p.
Thus, for $t > 0$,

$$\begin{aligned} &\left| \frac{f(x+t) + f(x-t) - 2f(x)}{t^2} \right| \\ &\qquad = \lim_{p \to \infty} \left| \frac{(f * K_p)(x+t) + (f * K_p)(x-t) - 2(f * K_p)(x)}{t^2} \right| \leqslant C. \end{aligned}$$

This shows that f'' is in L_∞ and, thus, $f' \in \mathrm{Lip}\ 1$; the theorem is proved.

5. Saturation of Optimal and Quasi-optimal Sequences

The following two theorems determine the saturation classes of optimal and quasi-optimal sequences.

THEOREM 5.1. *If* (L_n) *is optimal then it is saturated with order* (n^{-2}) *and* $S(L_n) = \{f : f' \in \mathrm{Lip}\ 1\}$.

THEOREM 5.2. *If* (L_n) *is quasi-optimal then it is saturated with order* (n^{-2}). *If* m *is the smallest positive integer satisfying* (3.1) *then*

$$S(L_n) = \{f : f' \in \text{Lip } 1 \text{ and } f \text{ has period } 2\pi/m\}.$$

Proof. Clearly, Theorem 5.2 contains as a special case Theorem 5.1, namely, when $m = 1$. We shall use the following lemma which is contained implicitly in Curtis [7].

LEMMA. *There is a constant* $C_0 > 0$ *such that for any integers* n *and* k, $n \geqslant k \geqslant 1$, *we have*

$$\int_{-\pi}^{\pi} \sin^2 \frac{kt}{2} T_n(t)\, dt \geqslant C_0 \frac{k^2}{n^2}, \tag{5.1}$$

whenever T_n *is a non-negative trigonometric polynomial of degree* $\leqslant n$ *with*

$$\frac{1}{\pi} \int_{-\pi}^{\pi} T_n(t)\, dt = 1.$$

Proof of the Lemma. It follows from Curtiss [7] that for $k \geqslant 1, m = 1, 2,...,$

$$E_{km}(|\sin kx|) \geqslant \frac{1}{4\pi m},$$

where $E_j(f)$ is the error in approximating f by trigonometric polynomials of degree $\leqslant j$, in the supremum norm. Given $n \geqslant k$, choose m so that

$$km \leqslant n < k(m + 1).$$

Then

$$E_n(|\sin kx|) \geqslant E_{k(m+1)}(|\sin kx|) \geqslant \frac{1}{4\pi(m + 1)}$$
$$= \frac{m}{m + 1} \frac{k}{4\pi km} \geqslant \frac{m}{m + 1} \frac{k}{4\pi n}.$$

Since $m \geqslant 1$, $(m/m + 1) \geqslant \frac{1}{2}$, so that

$$E_n(|\sin kx|) \geqslant \frac{k}{8\pi n}. \tag{5.2}$$

Now $\int_{-\pi}^{\pi} |\sin kt|\, T_n(t - x)\, dt$ is a trigonometric polynomial of degree $\leqslant n$, so that

$$\max_{x \in [\pi, -\pi]} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} (|\sin kx| - |\sin kt|)\, T_n(t - x)\, dt \right| \geqslant \frac{k}{8\pi n}. \tag{5.3}$$

However,

$$\begin{aligned}&\big|\,|\sin kt| - |\sin kx|\,\big|\\ &\quad\leqslant |\sin kt - \sin kx| = 2\left|\cos\frac{k}{2}(t+x)\right|\left|\sin k\frac{(t-x)}{2}\right|\\ &\quad\leqslant 2\left|\sin\frac{k(t-x)}{2}\right|,\end{aligned}$$

and so

$$\max_{x\in[\pi,-\pi]}\left|\frac{1}{\pi}\int_{-\pi}^{\pi}|\sin kt| - |\sin kx|\,T_n(t-x)\,dt\right| \leqslant \frac{2}{\pi}\int_{-\pi}^{\pi}\left|\sin\frac{kt}{2}\right| T_n(t)\,dt.$$

Using (5.3), we have

$$\int_{-\pi}^{\pi}\left|\sin\frac{kt}{2}\right| T_n(t)\,dt \geqslant \frac{\pi k}{16\pi n}. \tag{5.4}$$

Finally, using the Cauchy-Schwarz inequality for linear positive functionals, we have

$$\int_{-\pi}^{\pi}\left|\sin\frac{kt}{2}\right| T_n(t)\,dt \leqslant \left(\int_{-\pi}^{\pi} T_n(t)\,dt\right)^{1/2}\left(\int_{-\pi}^{\pi}\sin^2\frac{kt}{2}\,T_n(t)\,dt\right)^{1/2},$$

which, together with (5.4), gives (5.1), with $C_0 = 16^{-2}\pi^{-1}$; the proof of the Lemma is complete.

Now suppose (L_n) is quasi-optimal: $L_n(f) = f * T_n$. Let f_0 be a non-constant function for which

$$\|f_0 - L_n(f_0)\| = O\left(\frac{1}{n^2}\right).$$

There is an integer $m \neq 0$ such that $\hat{f}_0(m) \neq 0$. For this m,

$$\hat{f}_0(m)(1 - 2\hat{T}_n(m)) = O\left(\frac{1}{n^2}\right)$$

and, thus,

$$1 - 2\hat{T}_n(m) = O\left(\frac{1}{n^2}\right).$$

By virtue of (5.1),

$$\overline{\lim}\,\frac{1 - 2\hat{T}_n(k)}{1 - 2\hat{T}_n(m)} > 0, \qquad k = \pm 1, \pm 2, \ldots,$$

which, by Theorem 3.1, shows that (L_n) is saturated with order (n^{-2}).

If m is the smallest integer for which (3.1) is satisfied, we know from Theorem 3.1 that any function in $S(L_n)$ has period $2\pi/m$. We now choose a sequence

$$d\lambda_{n_j}(t) = n_j^2 \sin^2 \frac{mt}{2} T_{n_j}(t)$$

which converges weak *; say, to $d\lambda$. Because of Theorem 4.3, we need only show that $d\lambda$ has an atom at one of the zeros of $\sin^2 mt/2$ or, equivalently, that there is a constant $\alpha > 0$ such that

$$\lim_{\epsilon\to 0} \int_{S_\epsilon} d\lambda(t) \geqslant \alpha, \tag{5.5}$$

where $S_\epsilon = \bigcup_k (x_k - \epsilon, x_k + \epsilon)$, the x_k's being the zeros of $\sin^2 mt/2$.

We now proceed to show that (5.5) holds for $\alpha = \frac{1}{2}C_0m^2$, with C_0 the constant in (5.1). Let $0 < \epsilon < \pi/m$ and let $g_n(x) = 0$ on S_ϵ, $g_n(x) = n^2Tn(x)$ on $[-\pi, \pi] - S_\epsilon$, $n = 1, 2,\ldots$. Then, for some constant C_1 and for $n = 1,2,\ldots$,

$$\int_{-\pi}^{\pi} g_{n_j}(t)\, dt \leqslant \frac{1}{\sin^2 m\epsilon/2} \int_{-\pi}^{\pi} \sin^2 \frac{mt}{2} n_j^2 T_{n_j}(t)\, dt \leqslant C_1 .$$

Therefore, there is a subsequence of (n_j) (which for simplicity of notation we denote also by (n_j) and a measure $d\nu$ such that (g_{n_j}) converges weak * to $d\nu$. In particular,

$$\lim_{j\to\infty} \int_{-\pi}^{\pi} \sin^2 \frac{kt}{2} g_{n_j}(t)\, dt = \int_{-\pi}^{\pi} \sin^2 \frac{kt}{2} d\nu(t) \leqslant \int_{-\pi}^{\pi} |\, d\nu \,|.$$

Choose k_0 to be an integer so large that

$$\int_{-\pi}^{\pi} |\, d\nu \,| \leqslant \frac{1}{4} C_0 k_0^2.$$

Then, for j sufficiently large, we have, from (5.1), that

$$\begin{aligned}
&\int_{S_\epsilon} \sin^2 \left(\frac{k_0mt}{2}\right) T_{n_j}(t)\, dt \\
&\quad\geqslant \int_{-\pi}^{\pi} \sin^2 \left(\frac{k_0mt}{2}\right) T_{n_j}(t)\, dt - \frac{1}{n_j^2} \int_{-\pi}^{\pi} \sin^2 \left(\frac{k_0mt}{2}\right) g_{n_j}(t)\, dt \\
&\quad\geqslant \frac{C_0k_0^2m^2}{(n_j)^2} - \frac{1}{(n_j)^2} \int_{-\pi}^{\pi} |\, d\nu \,| \geqslant \frac{\frac{1}{2}C_0k_0^2m^2}{(n_j)^2} .
\end{aligned} \tag{5.3}$$

Finally, $\sin^2 (k_0 mt/2) \leqslant k_0^2 \sin^2 (mt/2)$ and, thus, from (5.3),

$$\int_{S_\epsilon} d\lambda_{n_j}(t) \geqslant \frac{1}{2} C_0 m^2$$

for j sufficiently large.

If we now take any function F in $C^*[-\pi, \pi]$, of norm one, which is one on S_ϵ and zero outside $S_{2\epsilon}$, we find

$$\int_{S_{2\epsilon}} d\lambda(t) \geqslant \lim_{j\to\infty} \int_{-\pi}^{\pi} F(x)\, d\lambda_{n_j}(t) \geqslant \overline{\lim_{j\to\infty}} \int_{S_\epsilon} d\lambda_{n_j}(t) \geqslant \frac{1}{2} C_0 m^2.$$

Since this estimate is independent of ϵ, the theorem is proved.

References

1. G. Sunouchi and C. Watari, On determination of the Saturation, *Proc. Japan Acad.*, **34** (1958), 477–481.
2. A. H. Tureckiĭ, On classes of saturation for certain methods of summation of Fourier series of continuous periodic functions, *Amer. Math. Soc. Transl.*, (2) **26** (1963), 263–272.
3. G. G. Lorentz, "Approximation of Functions," Holt, Rinehart and Winston, New York, 1968.
4. D. Jackson, "The Theory of Approximation," *Amer. Math. Soc.* Colloquium Publications, Vol. **XI**, New York, 1930.
5. P. P. Korovkin, Asymptotic properties of positive methods for summation of Fourier series, *Uspehi Mat. Nauk* **15** (1960), No. **1** (91), 207–212. (Russian).
6. P. P. Korovkin, Linear Operator and Approximation Theory, Hindustan Publishing Corp., Delhi, 1960.
7. P. C. Curtis, The degree of approximation by positive convolution operators, *Michigan Math. J.*, **12** (1965), 155–160.
8. O. Shisha and B. Mond, The degree of approximation to periodic functions by linear positive operators, *J. Approximation Theory*, **1** (1968), 335–339.
9. E. Görlich and E. L. Stark, A unified approach to three problems on approximation by positive linear operators, to appear in the Proceedings of the Colloquium on the Constructive Theory of Functions held in Budapest in 1969.
10. P. L. Butzer and E. Görlich, Saturationsklassen und asymptotische Eigenschaften trigonometrischer singulärer Integrale, Festschrift zur Gedächtnisfeier für Karl Weierstrass 1815–1965, Westdeutscher Verlag, 1967.
11. On approximation theory. Proceeding of the Conference held in the Mathematical Research Institute at Oberwolfach, 1963, Birkhauser, Basel, 1964.